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Proper resolutions and Gorenstein categories

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ABSTRACT

Let \mathcal{A} be an abelian category and \mathcal{C} an additive full subcategory of \mathcal{A} . We provide a method to construct a proper \mathcal{C} -resolution (resp. coproper \mathcal{C} -coresolution) of one term in a short exact sequence in \mathcal{A} from that of the other two terms. By using these constructions, we answer affirmatively an open question on the stability of the Gorenstein category $\mathcal{G}(\mathcal{C})$ posed by Sather-Wagstaff, Sharif and White; and also prove that $\mathcal{G}(\mathcal{C})$ is closed under direct summands. In addition, we obtain some criteria for computing the \mathcal{C} -dimension and the $\mathcal{G}(\mathcal{C})$ -dimension of an object in \mathcal{A} .

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1. Introduction

Auslander and Bridger generalized in [1] finitely generated projective modules to finitely generated modules of Gorenstein dimension zero over (commutative) Noetherian rings. Furthermore, Enochs and Jenda introduced in [10] Gorenstein projective modules for arbitrary modules over a general ring, which is a generalization of finitely generated modules of Gorenstein dimension zero. Also in [10], Gorenstein injective modules were introduced as the dual of Gorenstein projective modules. It is well known that the class of modules of Gorenstein dimension zero and that of Gorenstein projective modules coincide for finitely generated modules over a left and right Noetherian ring, and that Gorenstein projective modules and Gorenstein injective modules share many nice properties of projective modules and injective modules, respectively (cf. [10,11,14]). The homological properties of

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Gorenstein projective and injective modules and some related generalized versions have been studied by many authors, see [1,4,6–8,10–21], and the literatures listed in them.

Let \mathcal{A} be an abelian category and \mathcal{C} an additive full subcategory of \mathcal{A} . Sather-Wagstaff, Sharif and White introduced in [18] the Gorenstein category $\mathcal{G}(\mathcal{C})$, which unifies the following notions: modules of Gorenstein dimension zero [1], Gorenstein projective modules, Gorenstein injective modules [10], V -Gorenstein projective modules, V -Gorenstein injective modules [12], and so on. The Gorenstein subcategory $\mathcal{G}(\mathcal{C})$ of \mathcal{A} is defined as $\mathcal{G}(\mathcal{C}) = \{M \text{ is an object in } \mathcal{A} \mid \text{there exists an exact sequence } \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots \text{ in } \mathcal{C}, \text{ which is both } \text{Hom}_{\mathcal{A}}(\mathcal{C}, -)\text{-exact and } \text{Hom}_{\mathcal{A}}(-, \mathcal{C})\text{-exact, such that } M \cong \text{Im}(C_0 \rightarrow C^0)\}$. Set $\mathcal{G}^0(\mathcal{C}) = \mathcal{C}$, $\mathcal{G}^1(\mathcal{C}) = \mathcal{G}(\mathcal{C})$, and inductively set $\mathcal{G}^{n+1}(\mathcal{C}) = \mathcal{G}(\mathcal{G}^n(\mathcal{C}))$ for any $n \geq 1$. They proved that when \mathcal{C} is self-orthogonal, $\mathcal{G}(\mathcal{C})$ possesses many nice properties. For example, in this case, $\mathcal{G}(\mathcal{C})$ is closed under extensions and \mathcal{C} is a projective generator and an injective cogenerator for $\mathcal{G}(\mathcal{C})$, which induce that $\mathcal{G}^n(\mathcal{C}) = \mathcal{G}(\mathcal{C})$ for any $n \geq 1$. Also in this case, they proved that $\mathcal{G}(\mathcal{C})$ is closed under direct summands. In particular, they posed the following open question:

Question 1.1. (See [18, Question 5.8].) Must there be an equality $\mathcal{G}^n(\mathcal{C}) = \mathcal{G}(\mathcal{C})$ for any $n \geq 1$?

In this paper, we will prove that the answer to this question is affirmative. The used methods are to construct a certain proper resolution (resp. coproper coresolution) of one term in a short exact sequence from that of the other two terms. This paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

In Section 3, we provide a method to construct a proper resolution (resp. coproper coresolution) of the first or last term in a short exact sequence from that of the other two terms. We will prove the following two theorems and their dual results.

Theorem 1.2. Let \mathcal{C} be a full subcategory of an abelian category \mathcal{A} and $0 \rightarrow X \rightarrow X^0 \rightarrow X^1 \rightarrow 0$ an exact sequence in \mathcal{A} . Let

$$\cdots \rightarrow C_i^j \rightarrow \cdots \rightarrow C_1^j \rightarrow C_0^j \rightarrow X^j \rightarrow 0 \tag{1.1(j)}$$

be a proper \mathcal{C} -resolution of X^j for $j = 0, 1$. Then

(1) We get the following exact sequences:

$$\cdots \rightarrow C_{i+1}^1 \oplus C_i^0 \rightarrow \cdots \rightarrow C_2^1 \oplus C_1^0 \rightarrow C \rightarrow X \rightarrow 0 \tag{1.2}$$

and

$$0 \rightarrow C \rightarrow C_1^1 \oplus C_0^0 \rightarrow C_0^1 \rightarrow 0.$$

(2) If the exact sequence (1.1(j)) is $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact for $j = 0, 1$, then so is the exact sequence (1.2).

(3) If \mathcal{C} is closed under finite direct sums and kernels of epimorphisms, then the exact sequence (1.2) is a proper \mathcal{C} -resolution of X .

Theorem 1.3. Let \mathcal{C} be a full subcategory of an abelian category \mathcal{A} and

$$0 \rightarrow X_1 \rightarrow X_0 \rightarrow X \rightarrow 0 \tag{1.3}$$

an exact sequence in \mathcal{A} . Let

$$C_j^n \rightarrow \cdots \rightarrow C_j^1 \rightarrow C_j^0 \rightarrow X_j \rightarrow 0 \tag{1.4(j)}$$

be a proper \mathcal{C} -resolution of X_j for $j = 0, 1$. Then

(1) We get the following exact sequence:

$$C_0^n \oplus C_1^{n-1} \rightarrow \dots \rightarrow C_0^2 \oplus C_1^1 \rightarrow C_0^1 \oplus C_1^0 \rightarrow C_0^0 \rightarrow X \rightarrow 0. \tag{1.5}$$

- (2) If all the exact sequences (1.3) and (1.4(j)) are $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact for $j = 1, 2$, then so is the exact sequence (1.5).
- (3) If the exact sequence (1.3) is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact, then the exact sequence (1.5) is a proper \mathcal{C} -resolution of X .

As applications of these constructions, in Section 4 we prove the following result, in which the first assertion answers Question 1.1 affirmatively.

Theorem 1.4. *Let \mathcal{A} be an abelian category and \mathcal{C} an additive full subcategory of \mathcal{A} . Then we have*

- (1) $\mathcal{G}^n(\mathcal{C}) = \mathcal{G}(\mathcal{C})$ for any $n \geq 1$.
- (2) $\mathcal{G}(\mathcal{C})$ is closed under direct summands.

Let \mathcal{A} be an abelian category and \mathcal{C} a full subcategory of \mathcal{A} . For a positive integer n , an object A in \mathcal{A} is called an n - \mathcal{C} -syzygy object (of an object M) if there exists an exact sequence $0 \rightarrow A \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0$ in \mathcal{A} with all C_i objects in \mathcal{C} . For an object M in \mathcal{A} , $\mathcal{C}\text{-dim} M$ is defined as $\inf\{n \geq 0 \mid \text{there exists an exact sequence } 0 \rightarrow C_n \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0 \text{ in } \mathcal{A} \text{ with all } C_i \text{ objects in } \mathcal{C}\}$. We set $\mathcal{C}\text{-dim} M$ infinity if no such integer exists.

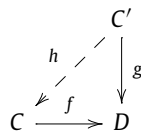
In Section 5, assume that \mathcal{C} is closed under extensions. We first prove that for a positive integer n , an object A in \mathcal{A} is n - \mathcal{C} -syzygy if and only if it is n -cogen \mathcal{C} -syzygy, where $\text{cogen } \mathcal{C}$ is a cogenerator for \mathcal{C} . Next we prove that if \mathcal{X} is a generator-cogenerator for \mathcal{C} , then for any object M in \mathcal{A} and $n \geq 0$, $\mathcal{C}\text{-dim} M \leq n$ if and only if for every non-negative integer t such that $0 \leq t \leq n$, there exists an exact sequence: $0 \rightarrow X_n \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$ in \mathcal{A} such that X_t is an object in \mathcal{C} and all X_i are objects in \mathcal{X} for $i \neq t$. As a consequence, when \mathcal{C} is self-orthogonal, we obtain some criteria for computing $\mathcal{G}(\mathcal{C})\text{-dim} M$ if it is finite.

2. Preliminaries

Throughout this paper, \mathcal{A} is an abelian category, all subcategories are full subcategories of \mathcal{A} closed under isomorphisms. We fix a subcategory \mathcal{C} of \mathcal{A} .

In this section, we give some terminology and some preliminary results.

Definition 2.1. (See [9].) Let $\mathcal{C} \subseteq \mathcal{D}$ be subcategories of \mathcal{A} . The morphism $f : C \rightarrow D$ in \mathcal{A} with C an object in \mathcal{C} and D an object in \mathcal{D} is called a \mathcal{C} -precover of D if for any morphism $g : C' \rightarrow D$ in \mathcal{A} with C' an object in \mathcal{C} , there exists a morphism $h : C' \rightarrow C$ such that the following diagram commutes:



The morphism $f : C \rightarrow D$ is called *right minimal* if a morphism $h : C \rightarrow C$ is an automorphism whenever $f = fh$. A \mathcal{C} -precover $f : C \rightarrow D$ is called a \mathcal{C} -cover if f is right minimal. Dually, the notions of a \mathcal{C} -preenvelope, a *left minimal morphism* and a \mathcal{C} -envelope are defined. Following Auslander and Reiten's

terminology in [2], a \mathcal{C} -(pre)cover and a \mathcal{C} -(pre)envelope are called a (minimal) right \mathcal{C} -approximation and a (minimal) left \mathcal{C} -approximation, respectively.

Recall that an exact sequence in \mathcal{A} is called $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact if it remains still exact after applying the functor $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$. Let M be an object in \mathcal{A} . An exact sequence (of finite or infinite length):

$$\dots \xrightarrow{f_{i+1}} C_i \xrightarrow{f_i} \dots \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} M \rightarrow 0$$

in \mathcal{A} with all C_i objects in \mathcal{C} is called a \mathcal{C} -resolution of M . Recall from [4] that the above exact sequence is called a proper \mathcal{C} -resolution of M if it is a \mathcal{C} -resolution of M and is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact, that is, each f_i is an epic \mathcal{C} -precover of $\text{Im } f_i$; and it is called a minimal proper \mathcal{C} -resolution of M if each f_i is an epic \mathcal{C} -cover of $\text{Im } f_i$. Dually, the notions of a $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact exact sequence, a \mathcal{C} -coresolution and a (minimal) coproper \mathcal{C} -coresolution of M are defined.

We now introduce the notion of strongly (co)proper (co)resolutions as follows.

Definition 2.2. Let M be an object in \mathcal{A} .

(1) A sequence (of finite or infinite length):

$$\dots \rightarrow X_i \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} is called strongly $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact if it is exact and $\text{Ext}_{\mathcal{A}}^1(\mathcal{C}, K_i) = 0$ for any $i \geq 1$, where $\text{Ext}_{\mathcal{A}}^1(\mathcal{C}, K_i) = \{\text{Ext}_{\mathcal{A}}^1(C, K_i) \mid C \text{ is an object in } \mathcal{C}\}$ and $K_i = \text{Im}(X_i \rightarrow X_{i-1})$. Dually, the notion of a strongly $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact exact sequence is defined.

(2) An exact sequence (of finite or infinite length):

$$\dots \rightarrow C_i \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} is called a strongly proper \mathcal{C} -resolution of M if it is a \mathcal{C} -resolution of M and is strongly $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact. Dually, the notion of a strongly coproper \mathcal{C} -coresolution of M is defined.

Remark 2.3. (1) It is easy to see that a strongly (co)proper \mathcal{C} -(co)resolution is a (co)proper \mathcal{C} -(co)resolution. But the converse does not hold true in general. For example, let \mathcal{C} be a full subcategory of \mathcal{A} closed under finite direct sums such that there exists an object M in \mathcal{C} with $\text{Ext}_{\mathcal{A}}^1(M, M) \neq 0$. Then the exact sequence:

$$0 \rightarrow M \xrightarrow{\begin{pmatrix} 1_M \\ 0 \end{pmatrix}} M \oplus M \xrightarrow{\begin{pmatrix} 0, 1_M \end{pmatrix}} M \rightarrow 0$$

is both a proper \mathcal{C} -resolution and a coproper \mathcal{C} -coresolution of M , but it neither a strongly proper \mathcal{C} -resolution nor a strongly coproper \mathcal{C} -coresolution of M .

(2) For a ring R , we use $\text{Mod } R$ to denote the category of left R -modules. We have that any projective resolution (resp. injective coresolution) of a left R -module M is just a strongly proper $\mathcal{P}(\text{Mod } R)$ -resolution (resp. strongly coproper $\mathcal{I}(\text{Mod } R)$ -coresolution) of M , where $\mathcal{P}(\text{Mod } R)$ (resp. $\mathcal{I}(\text{Mod } R)$) is the subcategory of $\text{Mod } R$ consisting of projective (resp. injective) modules.

(3) By Wakamatsu’s lemma (see [22, Lemma 2.1.1]), if \mathcal{C} is closed under extensions, then a minimal (co)proper \mathcal{C} -(co)resolution of an object M is a strongly (co)proper \mathcal{C} -(co)resolution of M .

The following two observations are useful in next section.

Lemma 2.4. *Let*

$$\begin{array}{ccc}
 M & \xrightarrow{f_1} & N \\
 g_1 \downarrow & & \downarrow g \\
 X & \xrightarrow{f} & Y
 \end{array}$$

be a commutative diagram in \mathcal{A} and D an object in \mathcal{A} .

- (1) *If this diagram is a pull-back diagram of f and g and if $\text{Hom}_{\mathcal{A}}(D, g)$ is epic, then $\text{Hom}_{\mathcal{A}}(D, g_1)$ is also epic.*
- (2) *If this diagram is a push-out diagram of f_1 and g_1 and if $\text{Hom}_{\mathcal{A}}(g_1, D)$ is epic, then $\text{Hom}_{\mathcal{A}}(g, D)$ is also epic.*

Proof. Assume that the given diagram is a pull-back diagram of f and g and that $\text{Hom}_{\mathcal{A}}(D, g)$ is epic. Let $\alpha \in \text{Hom}_{\mathcal{A}}(D, X)$. Then there exists $\beta \in \text{Hom}_{\mathcal{A}}(D, N)$ such that $f\alpha = \text{Hom}_{\mathcal{A}}(D, g)(\beta) = g\beta$. By the universal property of a pull-back diagram, there exists $\gamma \in \text{Hom}_{\mathcal{A}}(D, M)$ such that $\alpha = g_1\gamma = \text{Hom}_{\mathcal{A}}(D, g_1)(\gamma)$. So $\text{Hom}_{\mathcal{A}}(D, g_1)$ is epic and the assertion (1) follows.

Dually, we get the assertion (2). \square

Lemma 2.5.

- (1) *Let*

$$\begin{array}{ccc}
 K_1 & \xrightarrow{h} & K_0 \\
 \downarrow & & \downarrow \alpha_0 \\
 W_1 & \xrightarrow{g} & W_0 \\
 \downarrow & & \downarrow \\
 X_1 & \xrightarrow{f} & X_0 \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

be a commutative diagram in \mathcal{A} with exact columns and D an object in \mathcal{A} . If all of $\text{Hom}_{\mathcal{A}}(h, D)$, $\text{Hom}_{\mathcal{A}}(f, D)$ and $\text{Hom}_{\mathcal{A}}(\alpha_0, D)$ are epic, then so is $\text{Hom}_{\mathcal{A}}(g, D)$.

- (2) *Let*

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 K_1 & \xrightarrow{h} & K_0 \\
 \downarrow & & \downarrow \\
 W_1 & \xrightarrow{g} & W_0 \\
 \downarrow \beta_1 & & \downarrow \\
 X_1 & \xrightarrow{f} & X_0
 \end{array}$$

be a commutative diagram in \mathcal{A} with exact columns and D an object in \mathcal{A} . If all of $\text{Hom}_{\mathcal{A}}(D, h)$, $\text{Hom}_{\mathcal{A}}(D, f)$ and $\text{Hom}_{\mathcal{A}}(D, \beta_1)$ are epic, then so is $\text{Hom}_{\mathcal{A}}(D, g)$.

Proof. (1) By assumption, we get the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \uparrow \\
 & & & & & & \text{Hom}_{\mathcal{A}}(\alpha_0, D) \\
 & & & & & \uparrow & \\
 0 & \longleftarrow & \text{Hom}_{\mathcal{A}}(K_1, D) & \xleftarrow{\text{Hom}_{\mathcal{A}}(h, D)} & \text{Hom}_{\mathcal{A}}(K_0, D) & & \\
 & & \uparrow & & \uparrow & & \\
 & & \text{Hom}_{\mathcal{A}}(W_1, D) & \xleftarrow{\text{Hom}_{\mathcal{A}}(g, D)} & \text{Hom}_{\mathcal{A}}(W_0, D) & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longleftarrow & \text{Hom}_{\mathcal{A}}(X_1, D) & \xleftarrow{\text{Hom}_{\mathcal{A}}(f, D)} & \text{Hom}_{\mathcal{A}}(X_0, D) & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

By the snake lemma, $\text{Hom}_{\mathcal{A}}(g, D)$ is epic.

(2) It is dual to (1). \square

Definition 2.6. (See [18].) The Gorenstein subcategory $\mathcal{G}(\mathcal{C})$ of \mathcal{A} is defined as $\mathcal{G}(\mathcal{C}) = \{M \text{ is an object in } \mathcal{A} \mid \text{there exists an exact sequence:}$

$$\dots \rightarrow C_1 \rightarrow C_0 \rightarrow C^0 \rightarrow C^1 \rightarrow \dots \tag{2.1}$$

in \mathcal{C} , which is both $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact and $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact, such that $M \cong \text{Im}(C_0 \rightarrow C^0)$ }; in this case, (2.1) is called a complete \mathcal{C} -resolution of M .

Remark 2.7. (1) Let R be a left and right Noetherian ring and $\text{mod } R$ the category of finitely generated left R -modules. Put $\mathcal{P}(\text{mod } R)$ the subcategory of $\text{mod } R$ consisting of projective modules. Then $\mathcal{G}(\mathcal{P}(\text{mod } R))$ coincides with the subcategory of $\text{mod } R$ consisting of modules with Gorenstein dimension zero [1].

(2) For any ring R , $\mathcal{G}(\mathcal{P}(\text{Mod } R))$ (resp. $\mathcal{G}(\mathcal{I}(\text{Mod } R))$) coincides with the subcategory of $\text{Mod } R$ consisting of Gorenstein projective (resp. injective) modules [10].

(3) Let R be a left Noetherian ring, S a right Noetherian ring and ${}_R V_S$ a dualizing bimodule. Put $\mathcal{W} = \{V \otimes_S P \mid P \text{ is projective in } \text{Mod } S\}$ and $\mathcal{U} = \{\text{Hom}_S(V, E) \mid E \text{ is injective in } \text{Mod } S^{op}\}$. Then $\mathcal{G}(\mathcal{W})$ (resp. $\mathcal{G}(\mathcal{U})$) coincides with the subcategory of $\text{Mod } R$ consisting of V -Gorenstein projective (resp. injective) modules [12].

3. The constructions of (strongly) proper resolutions and coproper coresolutions

In this section we give a method to construct a (strongly) proper resolution (resp. coproper coresolution) of the first (resp. last) term in a short exact sequence from that of the other two terms, as well as give a method to construct a (strongly) proper resolution (resp. coproper coresolution) of the last (resp. first) term in a short exact sequence from that of the other two terms.

We first give the following easy observation, which is a generalization of the horseshoe lemma.

Lemma 3.1. Let $0 \rightarrow A \xrightarrow{f} A' \xrightarrow{g} A'' \rightarrow 0$ be an exact sequence in \mathcal{A} .

- (1) If there exist morphisms $\alpha \in \text{Hom}_{\mathcal{A}}(C, A)$, $\alpha'' \in \text{Hom}_{\mathcal{A}}(C'', A'')$ and $h \in \text{Hom}_{\mathcal{A}}(C'', A')$ such that $\alpha'' = gh$, then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & C & \xrightarrow{\begin{pmatrix} 1_C \\ 0 \end{pmatrix}} & C \oplus C'' & \xrightarrow{(0, 1_{C''})} & C'' & \longrightarrow & 0 \\
 & & \alpha \downarrow & & \alpha' \downarrow & & \alpha'' \downarrow & & \\
 0 & \longrightarrow & A & \xrightarrow{f} & A' & \xrightarrow{g} & A'' & \longrightarrow & 0
 \end{array}$$

where $\alpha' = (f\alpha, h) \in \text{Hom}_{\mathcal{A}}(C \oplus C'', A')$.

- (2) If there exist morphisms $\beta \in \text{Hom}_{\mathcal{A}}(A, D)$, $\beta'' \in \text{Hom}_{\mathcal{A}}(A'', D'')$ and $k \in \text{Hom}_{\mathcal{A}}(A', D)$ such that $\beta = kf$, then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & A & \xrightarrow{f} & A' & \xrightarrow{g} & A'' & \longrightarrow & 0 \\
 & & \beta \downarrow & & \beta' \downarrow & & \beta'' \downarrow & & \\
 0 & \longrightarrow & D & \xrightarrow{\begin{pmatrix} 1_D \\ 0 \end{pmatrix}} & D \oplus D'' & \xrightarrow{(0, 1_{D''})} & D'' & \longrightarrow & 0
 \end{array}$$

where $\beta' = \begin{pmatrix} k \\ \beta'' \end{pmatrix} \in \text{Hom}_{\mathcal{A}}(A', D \oplus D'')$.

The following result contains [Theorem 1.2](#), which provides a method to construct a (strongly) proper resolution of the first term in a short exact sequence from that of the last two terms.

Theorem 3.2. Let $0 \rightarrow X \rightarrow X^0 \rightarrow X^1 \rightarrow 0$ be an exact sequence in \mathcal{A} . Let

$$\dots \rightarrow C_i^0 \rightarrow \dots \rightarrow C_1^0 \rightarrow C_0^0 \rightarrow X^0 \rightarrow 0 \tag{3.1}$$

be a \mathcal{C} -resolution of X^0 , and let

$$\dots \rightarrow C_i^1 \rightarrow \dots \rightarrow C_1^1 \rightarrow C_0^1 \rightarrow X^1 \rightarrow 0 \tag{3.2}$$

be a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequence in \mathcal{A} . Then

- (1) We get the following exact sequences:

$$\dots \rightarrow C_{i+1}^1 \oplus C_i^0 \rightarrow \dots \rightarrow C_2^1 \oplus C_1^0 \rightarrow C \rightarrow X \rightarrow 0 \tag{3.3}$$

and

$$0 \rightarrow C \rightarrow C_1^1 \oplus C_0^0 \rightarrow C_0^1 \rightarrow 0. \tag{3.4}$$

- (2) For an object D in \mathcal{A} , if both the exact sequences (3.1) and (3.2) are $\text{Hom}_{\mathcal{A}}(-, D)$ -exact, then so is the exact sequence (3.3); in particular, if both the exact sequences (3.1) and (3.2) are $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact, then so is the exact sequence (3.3).
- (3) For an object D in \mathcal{A} , if both the exact sequences (3.1) and (3.2) are $\text{Hom}_{\mathcal{A}}(D, -)$ -exact, then so is the exact sequence (3.3); in particular, if the exact sequence (3.1) is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact, then so is the exact sequence (3.3).

- (4) For an object D in \mathcal{A} , if the given exact sequence $0 \rightarrow X \rightarrow X^0 \rightarrow X^1 \rightarrow 0$ is $\text{Hom}_{\mathcal{A}}(-, D)$ -exact (resp. $\text{Hom}_{\mathcal{A}}(D, -)$ -exact), then so is the exact sequence (3.4); in particular, if the given exact sequence $0 \rightarrow X \rightarrow X^0 \rightarrow X^1 \rightarrow 0$ is $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact (resp. $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact), then so is the exact sequence (3.4).

Assume that \mathcal{C} is closed under finite direct sums and kernels of epimorphisms. Then we have

- (5) If the exact sequence (3.2) is a \mathcal{C} -resolution of X^1 , then the exact sequence (3.3) is a \mathcal{C} -resolution of X .
 (6) If both the exact sequences (3.1) and (3.2) are proper \mathcal{C} -resolutions of X^0 and X^1 respectively, then the exact sequence (3.3) is a proper \mathcal{C} -resolution of X .
 (7) If both the exact sequences (3.1) and (3.2) are strongly proper \mathcal{C} -resolutions of X^0 and X^1 respectively, then the exact sequence (3.3) is a strongly proper \mathcal{C} -resolution of X .

Proof. (1) Put $K_i^0 = \text{Im}(C_i^0 \rightarrow C_{i-1}^0)$ and $K_i^1 = \text{Im}(C_i^1 \rightarrow C_{i-1}^1)$ for any $i \geq 1$. Consider the following pull-back diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & K_1^1 & \xlongequal{\quad} & K_1^1 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & X & \longrightarrow & M & \longrightarrow & C_0^1 \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X & \longrightarrow & X^0 & \longrightarrow & X^1 \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Because the third column in the above diagram is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact, so is the middle column by Lemma 2.4(1). Thus by Lemma 3.1(1) we get the following commutative diagram with exact columns and rows and the middle row splitting:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \dashrightarrow & K_2^1 & \dashrightarrow & W_1 & \dashrightarrow & K_1^0 \dashrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C_1^1 & \longrightarrow & C_1^1 \oplus C_0^0 & \longrightarrow & C_0^0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K_1^1 & \longrightarrow & M & \longrightarrow & X^0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where $W_1 = \text{Ker}(C_1^1 \oplus C_0^0 \rightarrow M)$. It is easy to verify that the upper row in the above diagram is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact.

On the one hand, we have the following pull-back diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & W_1 & \xlongequal{\quad} & W_1 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & C & \longrightarrow & C_1^1 \oplus C_0^0 & \longrightarrow & C_0^1 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & X & \longrightarrow & M & \longrightarrow & C_0^1 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

On the other hand, again by Lemma 3.1(1) we get the following commutative diagram with exact columns and rows and the middle row splitting:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \dashrightarrow & K_3^1 & \dashrightarrow & W_2 & \dashrightarrow & K_2^0 \dashrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C_2^1 & \longrightarrow & C_2^1 \oplus C_1^0 & \longrightarrow & C_1^0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K_2^1 & \longrightarrow & W_1 & \longrightarrow & K_1^0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where $W_2 = \text{Ker}(C_2^1 \oplus C_1^0 \rightarrow W_1)$ and the upper row in the above diagram is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact. Continuing this process, we get the desired exact sequences (3.3) and (3.4) with $W_i = \text{Im}(C_{i+1}^1 \oplus C_i^0 \rightarrow C_i^1 \oplus C_{i-1}^0)$ for any $i \geq 2$ and $W_1 = \text{Im}(C_2^1 \oplus C_1^0 \rightarrow C)$.

(2) Let D be an object in \mathcal{A} and both the exact sequences (3.1) and (3.2) $\text{Hom}_{\mathcal{A}}(-, D)$ -exact. Then both the first and third columns in the second diagram are $\text{Hom}_{\mathcal{A}}(-, D)$ -exact. It yields that both the first row and the middle column in this diagram are also $\text{Hom}_{\mathcal{A}}(-, D)$ -exact. Thus the first column in the third diagram is $\text{Hom}_{\mathcal{A}}(-, D)$ -exact. Also by assumption, the first and third columns in the last diagram are $\text{Hom}_{\mathcal{A}}(-, D)$ -exact. So the middle column in this diagram is also $\text{Hom}_{\mathcal{A}}(-, D)$ -exact. Finally, we deduce that the exact sequence (3.3) is $\text{Hom}_{\mathcal{A}}(-, D)$ -exact.

(3) Let D be an object in \mathcal{A} and both the exact sequences (3.1) and (3.2) $\text{Hom}_{\mathcal{A}}(D, -)$ -exact. Then both the middle column in the second diagram and the first column in the third diagram are

$\text{Hom}_{\mathcal{A}}(D, -)$ -exact; and moreover from the last diagram we get a $\text{Hom}_{\mathcal{A}}(D, -)$ - exact exact sequence:

$$\cdots \rightarrow C_{i+1}^1 \oplus C_i^0 \rightarrow \cdots \rightarrow C_2^1 \oplus C_1^0 \rightarrow W_1 \rightarrow 0.$$

Finally, we deduce that the exact sequence (3.3) is $\text{Hom}_{\mathcal{A}}(D, -)$ -exact.

(4) Let D be an object in \mathcal{A} . If the given exact sequence $0 \rightarrow X \rightarrow X^0 \rightarrow X^1 \rightarrow 0$ is $\text{Hom}_{\mathcal{A}}(-, D)$ -exact, then it is easy to check that the second row in the first diagram (that is, the third row in the third diagram) and the second row in the third diagram are $\text{Hom}_{\mathcal{A}}(-, D)$ -exact. If the given exact sequence $0 \rightarrow X \rightarrow X^0 \rightarrow X^1 \rightarrow 0$ is $\text{Hom}_{\mathcal{A}}(D, -)$ -exact. Then by Lemma 2.4(1), the second row in the first diagram (that is, the third row in the third diagram) and the second row in the third diagram are $\text{Hom}_{\mathcal{A}}(D, -)$ -exact.

(5) It follows from the assumption and the assertion (1).

(6) It follows from the assumption and the assertion (3).

(7) If both the exact sequences (3.1) and (3.2) are strongly proper \mathcal{C} -resolutions of X^0 and X^1 respectively, then $\text{Ext}_{\mathcal{A}}^1(\mathcal{C}, K_i^j) = 0$ for any $i \geq 1$ and $j = 0, 1$. By the proof of (1), we have an exact sequence:

$$0 \rightarrow K_{i+1}^1 \rightarrow W_i \rightarrow K_i^0 \rightarrow 0$$

for any $i \geq 1$. So $\text{Ext}_{\mathcal{A}}^1(\mathcal{C}, W_i) = 0$ for any $i \geq 1$, and hence the exact sequence (3.3) is a strongly proper \mathcal{C} -resolution of X . \square

Based on Theorem 3.2, by using induction on n it is not difficult to get the following

Corollary 3.3. *Let \mathcal{C} be closed under finite direct sums and kernels of epimorphisms, and let $0 \rightarrow X \rightarrow X^0 \rightarrow X^1 \rightarrow \cdots \rightarrow X^n \rightarrow 0$ be an exact sequence in \mathcal{A} . Assume that*

$$\cdots \rightarrow C_i^j \rightarrow \cdots \rightarrow C_1^j \rightarrow C_0^j \rightarrow X^j \rightarrow 0 \tag{3.5(j)}$$

is a (strongly) proper \mathcal{C} -resolution of X^j for any $0 \leq j \leq n$. Then we have

(1)

$$\cdots \rightarrow \bigoplus_{i=0}^n C_{i+3}^i \rightarrow \bigoplus_{i=0}^n C_{i+2}^i \rightarrow \bigoplus_{i=0}^n C_{i+1}^i \rightarrow C \rightarrow X \rightarrow 0 \tag{3.6}$$

is a (strongly) proper \mathcal{C} -resolution of X , and there exists an exact sequence:

$$0 \rightarrow C \rightarrow \bigoplus_{i=0}^n C_i^i \rightarrow \bigoplus_{i=1}^n C_{i-1}^i \rightarrow \bigoplus_{i=2}^n C_{i-2}^i \rightarrow \cdots \rightarrow C_0^{n-1} \oplus C_1^n \rightarrow C_0^n \rightarrow 0.$$

(2) If all (3.5(j)) are $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact, then so is (3.6).

The next two results are dual to Theorem 3.2 and Corollary 3.3 respectively. The following result provides a method to construct a (strongly) coproper coresolution of the last term in a short exact sequence from that of the first two terms.

Theorem 3.4. Let $0 \rightarrow Y_1 \rightarrow Y_0 \rightarrow Y \rightarrow 0$ be an exact sequence in \mathcal{A} . Let

$$0 \rightarrow Y_0 \rightarrow C_0^0 \rightarrow C_0^1 \rightarrow \dots \rightarrow C_0^i \rightarrow \dots \tag{3.7}$$

be a \mathcal{C} -coresolution of Y_0 , and let

$$0 \rightarrow Y_1 \rightarrow C_1^0 \rightarrow C_1^1 \rightarrow \dots \rightarrow C_1^i \rightarrow \dots \tag{3.8}$$

be a $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact exact sequence in \mathcal{A} . Then

(1) We get the following exact sequences:

$$0 \rightarrow Y \rightarrow C \rightarrow C_0^1 \oplus C_1^2 \rightarrow \dots \rightarrow C_0^i \oplus C_1^{i+1} \rightarrow \dots \tag{3.9}$$

and

$$0 \rightarrow C_1^0 \rightarrow C_0^0 \oplus C_1^1 \rightarrow C \rightarrow 0. \tag{3.10}$$

- (2) For an object D in \mathcal{A} , if both the exact sequences (3.7) and (3.8) are $\text{Hom}_{\mathcal{A}}(D, -)$ -exact, then so is the exact sequence (3.9); in particular, if both the exact sequences (3.7) and (3.8) are $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact, then so is the exact sequence (3.9).
- (3) For an object D in \mathcal{A} , if both the exact sequences (3.7) and (3.8) are $\text{Hom}_{\mathcal{A}}(-, D)$ -exact, then so is the exact sequence (3.9); in particular, if the exact sequence (3.7) is $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact, then so is the exact sequence (3.9).
- (4) For an object D in \mathcal{A} , if the given exact sequence $0 \rightarrow Y_1 \rightarrow Y_0 \rightarrow Y \rightarrow 0$ is $\text{Hom}_{\mathcal{A}}(D, -)$ -exact (resp. $\text{Hom}_{\mathcal{A}}(-, D)$ -exact), then so is the exact sequence (3.10); in particular, if the given exact sequence $0 \rightarrow Y_1 \rightarrow Y_0 \rightarrow Y \rightarrow 0$ is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact (resp. $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact), then so is the exact sequence (3.10).

Assume that \mathcal{C} is closed under finite direct sums and cokernels of monomorphisms. Then we have

- (5) If the exact sequence (3.8) is a \mathcal{C} -coresolution of Y_1 , then the exact sequence (3.9) is a \mathcal{C} -coresolution of Y .
- (6) If both the exact sequences (3.7) and (3.8) are coproper \mathcal{C} -coresolutions of Y_0 and Y_1 respectively, then the exact sequence (3.9) is a coproper \mathcal{C} -coresolution of Y .
- (7) If both the exact sequences (3.7) and (3.8) are strongly coproper \mathcal{C} -coresolutions of Y_0 and Y_1 respectively, then the exact sequence (3.9) is a strongly coproper \mathcal{C} -coresolution of Y .

Proof. It is completely dual to the proof of Theorem 3.2, so we omit it. \square

Based on Theorem 3.4, by using induction on n it is not difficult to get the following

Corollary 3.5. Let \mathcal{C} be closed under finite direct sums and cokernels of monomorphisms and let $0 \rightarrow Y_n \rightarrow \dots \rightarrow Y_1 \rightarrow Y_0 \rightarrow Y \rightarrow 0$ be an exact sequence in \mathcal{A} . Assume that

$$0 \rightarrow Y_j \rightarrow C_j^0 \rightarrow C_j^1 \rightarrow \dots \rightarrow C_j^i \rightarrow \dots \tag{3.11(j)}$$

is a (strongly) coproper \mathcal{C} -coresolution of Y_j for any $0 \leq j \leq n$. Then we have

(1)

$$0 \rightarrow Y \rightarrow C \rightarrow \bigoplus_{i=0}^n C_i^{i+1} \rightarrow \bigoplus_{i=0}^n C_i^{i+2} \rightarrow \bigoplus_{i=0}^n C_i^{i+3} \rightarrow \dots \tag{3.12}$$

is a (strongly) coproper \mathcal{C} -coresolution of Y , and there exists an exact sequence:

$$0 \rightarrow C_n^0 \rightarrow C_{n-1}^0 \oplus C_n^1 \rightarrow \dots \rightarrow \bigoplus_{i=2}^n C_i^{i-2} \rightarrow \bigoplus_{i=1}^n C_i^{i-1} \rightarrow \bigoplus_{i=0}^n C_i^i \rightarrow C \rightarrow 0.$$

(2) If all (3.11(j)) are $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact, then so is (3.12).

The following result contains **Theorem 1.3**, which provides a method to construct a (strongly) proper resolution of the last term in a short exact sequence from that of the first two terms.

Theorem 3.6. *Let*

$$0 \rightarrow X_1 \rightarrow X_0 \rightarrow X \rightarrow 0 \tag{3.13}$$

be an exact sequence in \mathcal{A} . Let

$$C_0^n \rightarrow \dots \rightarrow C_0^1 \rightarrow C_0^0 \rightarrow X_0 \rightarrow 0 \tag{3.14}$$

be a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequence and

$$C_1^{n-1} \rightarrow \dots \rightarrow C_1^1 \rightarrow C_1^0 \rightarrow X_1 \rightarrow 0 \tag{3.15}$$

a \mathcal{C} -resolution of X_1 in \mathcal{A} . Then

(1) We get the following exact sequence:

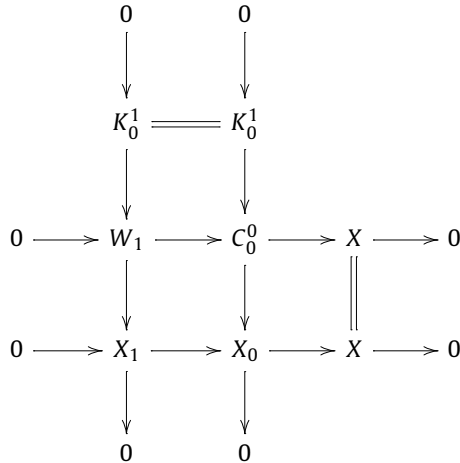
$$C_0^n \oplus C_1^{n-1} \rightarrow \dots \rightarrow C_0^2 \oplus C_1^1 \rightarrow C_0^1 \oplus C_1^0 \rightarrow C_0^0 \rightarrow X \rightarrow 0. \tag{3.16}$$

- (2) For an object D in \mathcal{A} , if all the exact sequences (3.13)–(3.15) are $\text{Hom}_{\mathcal{A}}(-, D)$ -exact, then so is the exact sequence (3.16); in particular, if all the exact sequences (3.13)–(3.15) are $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact, then so is the exact sequence (3.16).
- (3) For an object D in \mathcal{A} , if all the exact sequences (3.13)–(3.15) are $\text{Hom}_{\mathcal{A}}(D, -)$ -exact, then so is the exact sequence (3.16); in particular, if both the exact sequences (3.13) and (3.15) are $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact, then so is the exact sequence (3.16).

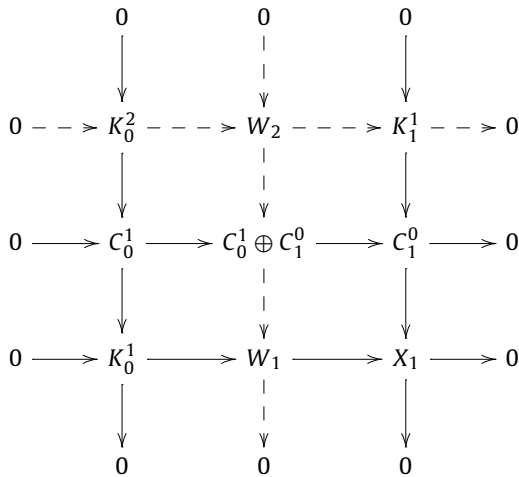
Assume that \mathcal{C} is closed under finite direct sums. Then we have

- (4) If the exact sequence (3.14) is a \mathcal{C} -resolution of X_0 , then the exact sequence (3.16) is a \mathcal{C} -resolution of X .
- (5) If the exact sequence (3.13) is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact and both the exact sequences (3.14) and (3.15) are proper \mathcal{C} -resolutions of X_0 and X_1 respectively, then the exact sequence (3.16) is a proper \mathcal{C} -resolution of X .
- (6) If the exact sequence (3.13) is strongly $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact and both the exact sequences (3.14) and (3.15) are strongly proper \mathcal{C} -resolutions of X_0 and X_1 respectively, then the exact sequence (3.16) is a strongly proper \mathcal{C} -resolution of X .

Proof. (1) Put $K_j^i = \text{Im}(C_j^i \rightarrow C_j^{i-1})$ for any $1 \leq i \leq n - j$ and $j = 0, 1$. Consider the following pull-back diagram:



Note that the middle column in the above diagram is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact. So by Lemma 2.4(1), the first column is also $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact. Then by Lemma 3.1(1) we get the following commutative diagram with exact columns and rows and the middle row splitting:



where $W_2 = \text{Ker}(C_0^1 \oplus C_1^0 \rightarrow W_1)$. It is easy to check that the upper row in the above diagram is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact. Then by using Lemma 3.1(1) iteratively we get the exact sequence (3.16) with $W_i = \text{Im}(C_0^i \oplus C_1^{i-1} \rightarrow C_0^{i-1} \oplus C_1^{i-2})$ for any $2 \leq i \leq n$ and $W_1 = \text{Im}(C_0^1 \oplus C_1^0 \rightarrow C_0^0)$.

(2) Let D be an object in \mathcal{A} and all the exact sequences (3.13)–(3.15) $\text{Hom}_{\mathcal{A}}(-, D)$ -exact. Then both the third row and the middle column in the first diagram are $\text{Hom}_{\mathcal{A}}(-, D)$ -exact. So the middle row in this diagram is $\text{Hom}_{\mathcal{A}}(-, D)$ -exact by Lemma 2.5(1). Both the first and third columns in the second diagram are $\text{Hom}_{\mathcal{A}}(-, D)$ -exact by assumption, so the middle column in this diagram is also $\text{Hom}_{\mathcal{A}}(-, D)$ -exact. Finally, we deduce that the exact sequence (3.16) is $\text{Hom}_{\mathcal{A}}(-, D)$ -exact.

(3) Let D be an object in \mathcal{A} and all the exact sequences (3.13)–(3.15) $\text{Hom}_{\mathcal{A}}(D, -)$ -exact. Then it is easy to check that both the middle row in the first diagram and the middle column in

the second diagram are $\text{Hom}_{\mathcal{A}}(D, -)$ -exact. Finally, we deduce that the exact sequence (3.16) is $\text{Hom}_{\mathcal{A}}(D, -)$ -exact.

(4) It follows from the assumption and the assertion (1).

(5) It follows from the assumption and the assertion (3).

(6) If the exact sequence (3.13) is strongly $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact and both the exact sequences (3.14) and (3.15) are strongly proper \mathcal{C} -resolutions of X_0 and X_1 respectively, then $\text{Ext}_{\mathcal{A}}^1(\mathcal{C}, K_j^i) = 0$ for any $1 \leq i \leq n - j$ and $j = 0, 1$. By the proof of (1), we have an exact sequence:

$$0 \rightarrow K_0^i \rightarrow W_i \rightarrow K_1^{i-1} \rightarrow 0$$

for any $1 \leq i \leq n$ (where $K_1^0 = X_1$). So $\text{Ext}_{\mathcal{A}}^1(\mathcal{C}, W_i) = 0$ for any $1 \leq i \leq n$, and hence the exact sequence (3.16) is a strongly proper \mathcal{C} -resolution of X . \square

Based on Theorem 3.6, by using induction on n it is not difficult to get the following

Corollary 3.7. *Let \mathcal{C} be closed under finite direct sums, and let*

$$X_n \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow X \rightarrow 0 \tag{3.17}$$

and

$$C_j^{n-j} \rightarrow \dots \rightarrow C_j^1 \rightarrow C_j^0 \rightarrow X_j \rightarrow 0 \tag{3.18(j)}$$

be exact sequences in \mathcal{A} for any $0 \leq j \leq n$.

(1) *Let the exact sequence (3.17) be $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact and (3.18(j)) a proper \mathcal{C} -resolution of X_j for any $0 \leq j \leq n$. Then*

$$\bigoplus_{i=0}^n C_i^{n-i} \rightarrow \bigoplus_{i=0}^{n-1} C_i^{(n-1)-i} \rightarrow \dots \rightarrow C_0^1 \oplus C_1^0 \rightarrow C_0^0 \rightarrow X \rightarrow 0 \tag{3.19}$$

is a proper \mathcal{C} -resolution of X ; furthermore, if (3.17) and all (3.18(j)) are $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact, then so is (3.19).

(2) *Let the exact sequence (3.17) be strongly $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact and (3.18(j)) a strongly proper \mathcal{C} -resolution of X_j for any $0 \leq j \leq n$. Then (3.19) is a strongly proper \mathcal{C} -resolution of X .*

The next two results are dual to Theorem 3.6 and Corollary 3.7 respectively. The following result provides a method to construct a (strongly) coproper coresolution of the first term in a short exact sequence from that of the last two terms.

Theorem 3.8. *Let*

$$0 \rightarrow Y \rightarrow Y^0 \rightarrow Y^1 \rightarrow 0 \tag{3.20}$$

be an exact sequence in \mathcal{A} . Let

$$0 \rightarrow Y^0 \rightarrow C_0^0 \rightarrow C_1^0 \rightarrow \dots \rightarrow C_n^0 \tag{3.21}$$

be a $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact exact sequence and

$$0 \rightarrow Y^1 \rightarrow C_0^1 \rightarrow C_1^1 \rightarrow \cdots \rightarrow C_{n-1}^1 \tag{3.22}$$

a \mathcal{C} -coresolution of Y^1 in \mathcal{A} . Then

(1) We get the following exact sequence:

$$0 \rightarrow Y \rightarrow C_0^0 \rightarrow C_0^1 \oplus C_1^0 \rightarrow C_1^1 \oplus C_2^0 \rightarrow \cdots \rightarrow C_{n-1}^1 \oplus C_n^0. \tag{3.23}$$

- (2) For an object D in \mathcal{A} , if all the exact sequences (3.20)–(3.22) are $\text{Hom}_{\mathcal{A}}(D, -)$ -exact, then so is the exact sequence (3.23); in particular, if all the exact sequences (3.20)–(3.22) are $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact, then so is the exact sequence (3.23).
- (3) For an object D in \mathcal{A} , if all the exact sequences (3.20)–(3.22) are $\text{Hom}_{\mathcal{A}}(-, D)$ -exact, then so is the exact sequence (3.23); in particular, if both the exact sequences (3.20) and (3.22) are $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact, then so is the exact sequence (3.23).

Assume that \mathcal{C} is closed under finite direct sums. Then we have

- (4) If the exact sequence (3.21) is a \mathcal{C} -coresolution of Y^0 , then the exact sequence (3.23) is a \mathcal{C} -coresolution of X .
- (5) If the exact sequence (3.20) is $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact and both the exact sequences (3.21) and (3.22) are coproper \mathcal{C} -coresolutions of Y^0 and Y^1 respectively, then the exact sequence (3.23) is a coproper \mathcal{C} -coresolution of Y .
- (6) If the exact sequence (3.20) is strongly $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact and both the exact sequences (3.21) and (3.22) are strongly coproper \mathcal{C} -coresolutions of Y^0 and Y^1 respectively, then the exact sequence (3.23) is a strongly coproper \mathcal{C} -coresolution of Y .

Proof. It is completely dual to the proof of Theorem 3.6, so we omit it. \square

Based on Theorem 3.8, by using induction on n it is not difficult to get the following

Corollary 3.9. Let \mathcal{C} be closed under finite direct sums, and let

$$0 \rightarrow Y \rightarrow Y^0 \rightarrow Y^1 \rightarrow \cdots \rightarrow Y^n \tag{3.24}$$

and

$$0 \rightarrow Y^j \rightarrow C_0^j \rightarrow C_1^j \rightarrow \cdots \rightarrow C_{n-j}^j \tag{3.25(j)}$$

be exact sequences in \mathcal{A} for any $0 \leq j \leq n$.

- (1) Let the exact sequence (3.24) be $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact and (3.25(j)) a coproper \mathcal{C} -coresolution of Y^j for any $0 \leq j \leq n$. Then

$$0 \rightarrow Y \rightarrow C_0^0 \rightarrow C_1^0 \oplus C_0^1 \rightarrow \cdots \rightarrow \bigoplus_{i=0}^{n-1} C_{(n-1)-i}^i \rightarrow \bigoplus_{i=0}^n C_{n-i}^i \tag{3.26}$$

is a coproper \mathcal{C} -coresolution of Y ; furthermore, if (3.24) and all (3.25(j)) are $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact, then so is (3.26).

- (2) Let the exact sequence (3.24) be strongly $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact and (3.25(j)) a strongly coproper \mathcal{C} -coresolution of Y^j for any $0 \leq j \leq n$. Then (3.26) is a strongly coproper \mathcal{C} -coresolution of Y .

4. Gorenstein categories

In the rest of this paper, all subcategories are additive subcategories of \mathcal{A} , that is, all subcategories are closed under finite direct sums. Set $\mathcal{G}^0(\mathcal{C}) = \mathcal{C}$, $\mathcal{G}^1(\mathcal{C}) = \mathcal{G}(\mathcal{C})$, and inductively set $\mathcal{G}^{n+1}(\mathcal{C}) = \mathcal{G}(\mathcal{G}^n(\mathcal{C}))$ for any $n \geq 1$. As an application of the results in the above section, we get the following result, which answers [Question 1.1](#) affirmatively.

Theorem 4.1. $\mathcal{G}^n(\mathcal{C}) = \mathcal{G}(\mathcal{C})$ for any $n \geq 1$.

Proof. It is easy to see that $\mathcal{C} \subseteq \mathcal{G}(\mathcal{C}) \subseteq \mathcal{G}^2(\mathcal{C}) \subseteq \mathcal{G}^3(\mathcal{C}) \subseteq \dots$ is an ascending chain of additive subcategories of \mathcal{A} .

Let M be an object in $\mathcal{G}^2(\mathcal{C})$ and

$$\dots \rightarrow G_1 \rightarrow G_0 \rightarrow G^0 \rightarrow G^1 \rightarrow \dots$$

a complete $\mathcal{G}(\mathcal{C})$ -resolution of M with $M \cong \text{Im}(G_0 \rightarrow G^0)$. Then for any $j \geq 0$, there exist exact sequences:

$$\dots \rightarrow C_j^i \rightarrow \dots \rightarrow C_j^1 \rightarrow C_j^0 \rightarrow G_j \rightarrow 0$$

and

$$0 \rightarrow G^j \rightarrow B_0^j \rightarrow B_1^j \rightarrow \dots \rightarrow B_i^j \rightarrow \dots$$

in \mathcal{A} with all C_j^i and B_i^j objects in \mathcal{C} , which are both $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact and $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact. By [Corollaries 3.7 and 3.9](#), we get exact sequences:

$$\dots \rightarrow \bigoplus_{i=0}^n C_i^{n-i} \rightarrow \dots \rightarrow C_0^1 \oplus C_1^0 \rightarrow C_0^0 \rightarrow M \rightarrow 0$$

and

$$0 \rightarrow M \rightarrow B_0^0 \rightarrow B_1^0 \oplus B_0^1 \rightarrow \dots \rightarrow \bigoplus_{i=0}^n B_{n-i}^i \rightarrow \dots$$

which are both $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact and $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact. So

$$\dots \rightarrow \bigoplus_{i=0}^n C_i^{n-i} \rightarrow \dots \rightarrow C_0^1 \oplus C_1^0 \rightarrow C_0^0 \rightarrow B_0^0 \rightarrow B_1^0 \oplus B_0^1 \rightarrow \dots \rightarrow \bigoplus_{i=0}^n B_{n-i}^i \rightarrow \dots$$

is a complete \mathcal{C} -resolution of M with $M \cong \text{Im}(C_0^0 \rightarrow B_0^0)$, and hence M is an object in $\mathcal{G}(\mathcal{C})$ and $\mathcal{G}^2(\mathcal{C}) \subseteq \mathcal{G}(\mathcal{C})$. Thus we have that $\mathcal{G}^2(\mathcal{C}) = \mathcal{G}(\mathcal{C})$. By using induction on n we get easily the assertion. \square

We remark that the inclusion $\mathcal{C} \subseteq \mathcal{G}(\mathcal{C})$ is strict in general. For example, let Λ be the finite-dimensional algebra over a field given by the quiver:

$$1 \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\alpha_2} \end{array} 2$$

modulo the ideal generated by $\{\alpha_1\alpha_2, \alpha_2\alpha_1\}$. Then Λ is a self-injective algebra with infinite global dimension. We use $S(1)$ (resp. $S(2)$) and $P(1)$ (resp. $P(2)$) to denote the simple Λ -module and the indecomposable projective Λ -module corresponding to the vertex 1 (resp. 2), respectively. Then

$$\dots \rightarrow P(1) \rightarrow P(2) \rightarrow P(1) \rightarrow S(1) \rightarrow 0$$

and

$$\dots \rightarrow P(2) \rightarrow P(1) \rightarrow P(2) \rightarrow S(2) \rightarrow 0$$

are the minimal projective resolutions of $S(1)$ and $S(2)$, respectively. So neither $S(1)$ nor $S(2)$ are in $\mathcal{P}(\Lambda)$ (the category consists of finitely generated projective Λ -modules). However, it is easy to see that both $S(1)$ and $S(2)$ are in $\mathcal{G}(\mathcal{P}(\Lambda))$.

Definition 4.2. (Cf. [18].) A subcategory \mathcal{X} of \mathcal{C} is called a *generator* (resp. *cogenerator*) for \mathcal{C} if for any object C in \mathcal{C} , there exists an exact sequence $0 \rightarrow C' \rightarrow X \rightarrow C \rightarrow 0$ (resp. $0 \rightarrow C \rightarrow X \rightarrow C' \rightarrow 0$) in \mathcal{C} with X an object in \mathcal{X} ; and \mathcal{X} is called a *projective generator* (resp. an *injective cogenerator*) for \mathcal{C} if \mathcal{X} is a generator (resp. cogenerator) for \mathcal{C} and $\text{Ext}_{\mathcal{A}}^i(X, C) = 0$ (resp. $\text{Ext}_{\mathcal{A}}^i(C, X) = 0$) for any object X in \mathcal{X} , any object C in \mathcal{C} and $i \geq 1$.

As an immediate consequence of Theorem 4.1, we get the following three corollaries. The first one generalizes [18, Proposition 4.6] and answers positively a question in [18, p. 492].

Corollary 4.3.

- (1) If \mathcal{X} is a (projective) generator for \mathcal{C} , then \mathcal{X} is a (projective) generator for $\mathcal{G}^n(\mathcal{C})$.
- (2) If \mathcal{X} is an (injective) cogenerator for \mathcal{C} , then \mathcal{X} is an (injective) cogenerator for $\mathcal{G}^n(\mathcal{C})$.

We define $\text{res } \tilde{\mathcal{X}} = \{M \text{ is an object in } \mathcal{A} \mid \text{there exists a Hom}_{\mathcal{A}}(\mathcal{C}, -)\text{-exact exact sequence } \dots \rightarrow C_i \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0 \text{ in } \mathcal{A} \text{ with all } C_i \text{ objects in } \mathcal{C}\}$. Dually, we define $\text{cores } \tilde{\mathcal{X}} = \{M \text{ is an object in } \mathcal{A} \mid \text{there exists a Hom}_{\mathcal{A}}(-, \mathcal{C})\text{-exact exact sequence } 0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \dots \rightarrow C^i \rightarrow \dots \text{ in } \mathcal{A} \text{ with all } C^i \text{ objects in } \mathcal{C}\}$ (see [18]).

The next corollary generalizes [18, Theorem 4.8].

Corollary 4.4. Let \mathcal{X} be a subcategory of \mathcal{C} . Then we have

- (1) If $\mathcal{C} \subseteq \text{res } \tilde{\mathcal{X}}$, then $\mathcal{G}^n(\mathcal{C}) \subseteq \text{res } \tilde{\mathcal{X}}$ for any $n \geq 0$.
- (2) If $\mathcal{C} \subseteq \text{cores } \tilde{\mathcal{X}}$, then $\mathcal{G}^n(\mathcal{C}) \subseteq \text{cores } \tilde{\mathcal{X}}$ for any $n \geq 0$.

Proof. (1) Let $\mathcal{C} \subseteq \text{res } \tilde{\mathcal{X}}$. Because \mathcal{X} is closed under finite sums, $\mathcal{G}(\mathcal{C}) \subseteq \text{res } \tilde{\mathcal{X}}$ by Corollary 3.7. Then the assertion follows from Theorem 4.1.

(2) It is dual to (1). □

We also have the following corollary, which generalizes [18, Theorem 4.9].

Corollary 4.5. Let \mathcal{X} be a subcategory of \mathcal{C} . If $\mathcal{C} \subseteq \text{res } \tilde{\mathcal{X}} \cap \text{cores } \tilde{\mathcal{X}}$, then $\mathcal{G}^n(\mathcal{C}) \subseteq \mathcal{G}(\mathcal{X})$ for any $n \geq 1$.

Proof. By using an argument similar to that in the proof of [Theorem 4.1](#), we get $\mathcal{G}(\mathcal{C}) \subseteq \mathcal{G}(\mathcal{X})$. Then the assertion follows from [Theorem 4.1](#). \square

As another application of the results in the above section, we get the following result, in which the second assertion shows that the assumption of the self-orthogonality of \mathcal{C} in [[18, Proposition 4.11](#)] is superfluous.

Theorem 4.6.

- (1) Both $\text{res } \tilde{\mathcal{C}}$ and $\text{cores } \tilde{\mathcal{C}}$ are closed under direct summands.
- (2) $\mathcal{G}(\mathcal{C})$ is closed under direct summands.

Proof. Assume that $M = X \oplus Y$ and

$$0 \rightarrow Y \xrightarrow{\begin{pmatrix} 0 \\ 1_Y \end{pmatrix}} M \xrightarrow{(1_X, 0)} X \rightarrow 0$$

is an exact and split sequence.

(1) We only prove $\text{res } \tilde{\mathcal{C}}$ is closed under direct summands. Dually, we get that $\text{cores } \tilde{\mathcal{C}}$ is also closed under direct summands.

Let M be an object in $\text{res } \tilde{\mathcal{C}}$ and

$$\dots \xrightarrow{f_{i+1}} C_i \xrightarrow{f_i} \dots \xrightarrow{f_3} C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} M \rightarrow 0$$

a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequence in \mathcal{A} with all C_i objects in \mathcal{C} . Then

$$C_0 \xrightarrow{(1_X, 0)f_0} X \rightarrow 0$$

is a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequence. Similarly,

$$C_0 \xrightarrow{(0, 1_Y)f_0} Y \rightarrow 0$$

is a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequence. By [Theorem 3.6](#), we get the following $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequences:

$$C_0 \oplus C_1 \rightarrow C_0 \rightarrow X \rightarrow 0$$

and

$$C_0 \oplus C_1 \rightarrow C_0 \rightarrow Y \rightarrow 0.$$

Again by [Theorem 3.6](#), we get the following $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequences:

$$C_0 \oplus C_1 \oplus C_2 \rightarrow C_0 \oplus C_1 \rightarrow C_0 \rightarrow X \rightarrow 0$$

and

$$C_0 \oplus C_1 \oplus C_2 \rightarrow C_0 \oplus C_1 \rightarrow C_0 \rightarrow Y \rightarrow 0.$$

Continuing this procedure, we finally get the following $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequences:

$$\cdots \rightarrow \bigoplus_{i=0}^{n-1} C_i \rightarrow \cdots \rightarrow C_0 \oplus C_1 \oplus C_2 \rightarrow C_0 \oplus C_1 \rightarrow C_0 \rightarrow X \rightarrow 0$$

and

$$\cdots \rightarrow \bigoplus_{i=0}^{n-1} C_i \rightarrow \cdots \rightarrow C_0 \oplus C_1 \oplus C_2 \rightarrow C_0 \oplus C_1 \rightarrow C_0 \rightarrow Y \rightarrow 0,$$

which implies that both X and Y are objects in $\text{res } \tilde{\mathcal{C}}$.

(2) Let M be an object in $\mathcal{G}(\mathcal{C})$ and

$$\cdots \rightarrow C_1 \rightarrow C_0 \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots$$

a complete \mathcal{C} -resolution of M with $M \cong \text{Im}(C_0 \rightarrow C^0)$. By (1) and [Theorem 3.6](#), we get the following exact sequence:

$$\cdots \rightarrow \bigoplus_{i=0}^{n-1} C_i \rightarrow \cdots \rightarrow C_0 \oplus C_1 \oplus C_2 \rightarrow C_0 \oplus C_1 \rightarrow C_0 \rightarrow X \rightarrow 0 \tag{4.1}$$

which is both $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact and $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact. Dually, we get the following exact sequence:

$$0 \rightarrow X \rightarrow C^0 \rightarrow C^0 \oplus C^1 \rightarrow C^0 \oplus C^1 \oplus C^2 \rightarrow \bigoplus_{i=0}^{n-1} C^i \rightarrow \cdots \tag{4.2}$$

which is also both $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact and $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact. Combining the sequences [\(4.1\)](#) with [\(4.2\)](#), we conclude that X is an object in $\mathcal{G}(\mathcal{C})$. \square

We also have the following two out of three property.

Proposition 4.7.

- (1) If \mathcal{C} is closed under kernels of epimorphisms, then so is $\text{res } \tilde{\mathcal{C}}$.
- (2) If \mathcal{C} is closed under cokernels of monomorphisms, then so is $\text{cores } \tilde{\mathcal{C}}$.

Let

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \tag{4.3}$$

be an exact sequence in \mathcal{A} .

- (3) If the exact sequence [\(4.3\)](#) is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact and X, Y are objects in $\text{res } \tilde{\mathcal{C}}$, then Z is also an object in $\text{res } \tilde{\mathcal{C}}$.
- (4) If the exact sequence [\(4.3\)](#) is $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact and Y, Z are objects in $\text{cores } \tilde{\mathcal{C}}$, then X is also an object in $\text{cores } \tilde{\mathcal{C}}$.
- (5) If the exact sequence [\(4.3\)](#) is both $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact and $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact, and if any two of X, Y and Z are objects in $\mathcal{G}(\mathcal{C})$, then the third term is also an object in $\mathcal{G}(\mathcal{C})$.

Proof. We get (1) and (2) by [Theorems 3.2 and 3.4](#) respectively. Dually, we get (3) and (4) by [Theorems 3.6 and 3.8](#) respectively.

(5) If X, Z are objects in $\mathcal{G}(\mathcal{C})$, then so is Y by [\[18, Proposition 4.4\]](#). By the above arguments, we get the other two assertions. \square

5. Gorenstein syzygies and dimension

In this section, we fix subcategories \mathcal{C} and \mathcal{X} of \mathcal{A} . We use $\text{gen } \mathcal{C}$ (resp. $\text{cogen } \mathcal{C}$) to denote a generator (resp. cogenerator) for \mathcal{C} . The following result plays a crucial role in this section.

Proposition 5.1. *Let \mathcal{C} be closed under extensions and*

$$0 \rightarrow A \rightarrow C_1 \xrightarrow{f} C_0 \rightarrow M \rightarrow 0 \tag{5.1}$$

an exact sequence in \mathcal{A} with C_0 and C_1 objects in \mathcal{C} .

(1) Then we have the following exact sequences:

$$0 \rightarrow A \rightarrow C'_1 \rightarrow P_0 \rightarrow M \rightarrow 0 \tag{5.2}$$

and

$$0 \rightarrow A \rightarrow I_1 \rightarrow C'_0 \rightarrow M \rightarrow 0 \tag{5.3}$$

in \mathcal{A} with C'_1, C'_0 objects in \mathcal{C} , P_0 an object in $\text{gen } \mathcal{C}$ and I_1 an object in $\text{cogen } \mathcal{C}$.

(2) Let D be an object in \mathcal{A} such that any short exact sequence in \mathcal{C} is $\text{Hom}_{\mathcal{A}}(D, -)$ -exact (resp. $\text{Hom}_{\mathcal{A}}(-, D)$ -exact). If (5.1) is $\text{Hom}_{\mathcal{A}}(D, -)$ -exact (resp. $\text{Hom}_{\mathcal{A}}(-, D)$ -exact), then so are both (5.2) and (5.3).

Proof. (1) There exists an exact sequence:

$$0 \rightarrow C'_0 \rightarrow P_0 \rightarrow C_0 \rightarrow 0$$

in \mathcal{A} with P_0 an object in $\text{gen } \mathcal{C}$ and C'_0 an object in \mathcal{C} . Then we have the following pull-back diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & C'_0 & \xlongequal{\quad} & C'_0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & N & \longrightarrow & P_0 & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \text{Im } f & \longrightarrow & C_0 & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Consider the following pull-back diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & C'_0 & \xlongequal{\quad} & C'_0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A & \longrightarrow & C'_1 & \longrightarrow & N \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & C_1 & \longrightarrow & \text{Im } f \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Because \mathcal{C} is closed under extensions and both C'_0 and C_1 are objects in \mathcal{C} , C'_1 is also an object in \mathcal{C} . Connecting the middle rows in the above two diagrams, then we get the first desired exact sequence. Dually, taking push-out, we get the second desired sequence.

(2) Let D be an object in \mathcal{A} such that any short exact sequence in \mathcal{C} is $\text{Hom}_{\mathcal{A}}(D, -)$ -exact (resp. $\text{Hom}_{\mathcal{A}}(-, D)$ -exact). Then the middle columns in the above two diagrams are $\text{Hom}_{\mathcal{A}}(D, -)$ -exact (resp. $\text{Hom}_{\mathcal{A}}(-, D)$ -exact). If (5.1) is $\text{Hom}_{\mathcal{A}}(D, -)$ -exact (resp. $\text{Hom}_{\mathcal{A}}(-, D)$ -exact), then both the third rows in the above two diagrams are $\text{Hom}_{\mathcal{A}}(D, -)$ -exact (resp. $\text{Hom}_{\mathcal{A}}(-, D)$ -exact). So both the middle rows in these two diagrams are also $\text{Hom}_{\mathcal{A}}(D, -)$ -exact (resp. $\text{Hom}_{\mathcal{A}}(-, D)$ -exact), and hence (5.2) is $\text{Hom}_{\mathcal{A}}(D, -)$ -exact (resp. $\text{Hom}_{\mathcal{A}}(-, D)$ -exact). Dually, one gets the other assertion. \square

Definition 5.2. Let n be a positive integer. If there exists an exact sequence $0 \rightarrow A \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \dots \rightarrow C_0 \rightarrow M \rightarrow 0$ in \mathcal{A} with all C_i objects in \mathcal{C} , then A is called an n - \mathcal{C} -syzygy object (of M), and M is called an n - \mathcal{C} -cosyzygy object (of A).

The following theorem is one of main results in this section.

Theorem 5.3. Let \mathcal{C} be closed under extensions, and let $n \geq 1$ and

$$0 \rightarrow A \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \dots \rightarrow C_0 \rightarrow M \rightarrow 0 \tag{5.4}$$

be an exact sequence in \mathcal{A} with all C_i objects in \mathcal{C} . Then we have the following

(1) There exist exact sequences:

$$0 \rightarrow A \rightarrow I_{n-1} \rightarrow I_{n-2} \rightarrow \dots \rightarrow I_0 \rightarrow N \rightarrow 0 \tag{5.5}$$

and

$$0 \rightarrow M \rightarrow N \rightarrow X \rightarrow 0 \tag{5.6}$$

in \mathcal{A} with all I_i objects in $\text{cogen } \mathcal{C}$ and X an object in \mathcal{C} . In particular, an object in \mathcal{A} is an n - \mathcal{C} -syzygy if and only if it is an n - $\text{cogen } \mathcal{C}$ -syzygy.

(2) There exist exact sequences:

$$0 \rightarrow B \rightarrow P_{n-1} \rightarrow P_{n-2} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0 \tag{5.7}$$

and

$$0 \rightarrow Y \rightarrow B \rightarrow A \rightarrow 0 \tag{5.8}$$

in \mathcal{A} with all P_i objects in $\text{gen } \mathcal{C}$ and Y an object in \mathcal{C} . In particular, an object in \mathcal{A} is an n - \mathcal{C} -cosyzygy if and only if it is an n - $\text{gen } \mathcal{C}$ -cosyzygy.

(3) Let D be an object in \mathcal{A} such that any short exact sequence in \mathcal{C} is $\text{Hom}_{\mathcal{A}}(D, -)$ -exact (resp. $\text{Hom}_{\mathcal{A}}(-, D)$ -exact). If (5.4) is $\text{Hom}_{\mathcal{A}}(D, -)$ -exact (resp. $\text{Hom}_{\mathcal{A}}(-, D)$ -exact), then so are all (5.5)–(5.8).

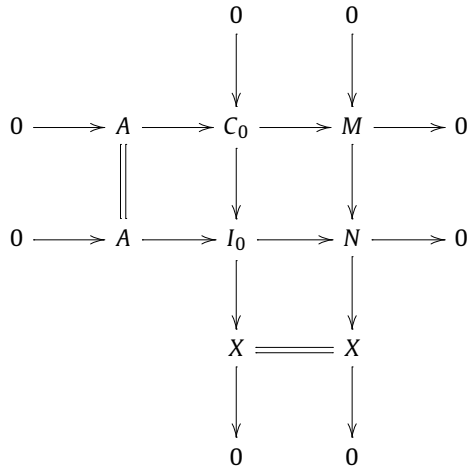
Proof. (1) We proceed by induction on n .
Let $n = 1$ and

$$0 \rightarrow A \rightarrow C_0 \rightarrow M \rightarrow 0$$

be an exact sequence in \mathcal{A} with C_0 an object in \mathcal{C} . Then there exists an exact sequence:

$$0 \rightarrow C_0 \rightarrow I_0 \rightarrow X \rightarrow 0$$

in \mathcal{A} with I_0 an object in $\text{cogen } \mathcal{C}$ and X an object in \mathcal{C} .
Consider the following push-out diagram:



Then the middle row and the third column in the above diagram are the desired exact sequences.
Now suppose that $n \geq 2$ and

$$0 \rightarrow A \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \cdots \rightarrow C_0 \rightarrow M \rightarrow 0$$

is an exact sequence in \mathcal{A} with all C_i objects in \mathcal{C} . Put $K = \text{Coker}(C_{n-1} \rightarrow C_{n-2})$. By Proposition 5.1(1), we get an exact sequence:

$$0 \rightarrow A \rightarrow I_{n-1} \rightarrow C'_{n-2} \rightarrow K \rightarrow 0$$

in \mathcal{A} with I_{n-1} an object in $\text{cogen } \mathcal{C}$ and C'_{n-2} an object in \mathcal{C} . Put $A' = \text{Im}(I_{n-1} \rightarrow C'_{n-2})$. Then we get an exact sequence:

$$0 \rightarrow A' \rightarrow C'_{n-2} \rightarrow C_{n-3} \rightarrow \dots \rightarrow C_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} . Now we get the assertion by the induction hypothesis.

- (2) The proof is dual to that of (1).
- (3) It follows from Proposition 5.1(2). \square

For any $n \geq 1$, we denote by $\Omega^n_{\mathcal{C}}(\mathcal{A})$ (resp. $\Omega^{-n}_{\mathcal{C}}(\mathcal{A})$) the subcategory of \mathcal{A} consisting of n - \mathcal{C} -syzygy (resp. n - \mathcal{C} -cosyzygy) objects.

Corollary 5.4. *Let \mathcal{C} be closed under extensions. Then for any $n \geq 1$ we have*

- (1) If \mathcal{X} is a cogenerator for \mathcal{C} , then $\Omega^n_{\mathcal{C}}(\mathcal{A}) = \Omega^n_{\mathcal{X}}(\mathcal{A})$.
- (2) If \mathcal{X} is a generator for \mathcal{C} , then $\Omega^{-n}_{\mathcal{C}}(\mathcal{A}) = \Omega^{-n}_{\mathcal{X}}(\mathcal{A})$.

For an object M in \mathcal{A} , the \mathcal{C} -dimension of M , denoted by $\mathcal{C}\text{-dim } M$, is defined as $\inf\{n \geq 0 \mid \text{there exists an exact sequence } 0 \rightarrow C_n \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0 \text{ in } \mathcal{A} \text{ with all } C_i \text{ objects in } \mathcal{C}\}$. We set $\mathcal{C}\text{-dim } M$ infinity if no such integer exists. A subcategory \mathcal{X} of \mathcal{C} is called a *generator-cogenerator* for \mathcal{C} if it is both a generator and a cogenerator for \mathcal{C} .

Another main result in this section is the following

Theorem 5.5. *Let \mathcal{C} be closed under extensions and \mathcal{X} a generator-cogenerator for \mathcal{C} . Then the following statements are equivalent for any object M in \mathcal{A} and $n \geq 0$.*

- (1) $\mathcal{C}\text{-dim } M \leq n$.
- (2) There exists an exact sequence:

$$0 \rightarrow C_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} with C_n an object in \mathcal{C} and all X_i objects in \mathcal{X} .

- (3) There exists an exact sequence:

$$0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow C_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} with C_0 an object in \mathcal{C} and all X_i objects in \mathcal{X} .

- (4) For every non-negative integer t such that $0 \leq t \leq n$, there exists an exact sequence:

$$0 \rightarrow X_n \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} such that X_t is an object in \mathcal{C} and all X_i are objects in \mathcal{X} for $i \neq t$.

Proof. (4) \Rightarrow (2) \Rightarrow (1) and (4) \Rightarrow (3) \Rightarrow (1) are trivial.

(1) \Rightarrow (4) We proceed by induction on n .

Let $\mathcal{C}\text{-dim } M \leq 1$ and

$$0 \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0$$

be an exact sequence in \mathcal{A} with C_0, C_1 objects in \mathcal{C} . By Proposition 5.1(1) with $A = 0$, we get the exact sequences

$$0 \rightarrow C'_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

and

$$0 \rightarrow X_1 \rightarrow C'_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} with X_0, X_1 objects in \mathcal{X} and C'_0, C'_1 objects in \mathcal{C} .

Now suppose $\mathcal{C}\text{-dim } M = n \geq 2$ and

$$0 \rightarrow C_n \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0$$

is an exact sequence in \mathcal{A} with all C_i objects in \mathcal{C} . Set $A = \text{Coker}(C_3 \rightarrow C_2)$. By applying Proposition 5.1(1) to the exact sequence:

$$0 \rightarrow A \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0,$$

we get the following exact sequences:

$$0 \rightarrow A \rightarrow C'_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

and

$$0 \rightarrow C_n \rightarrow \dots \rightarrow C_2 \rightarrow C'_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} with C'_1 an object in \mathcal{C} and X_0 an object in \mathcal{X} . Set $N = \text{Coker}(C_2 \rightarrow C'_1)$. Then we have $\mathcal{C}\text{-dim } M \leq n - 1$. By the induction hypothesis, there exists an exact sequence:

$$0 \rightarrow X_n \rightarrow \dots \rightarrow X_t \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} such that X_t is an object in \mathcal{C} and X_i is an object in \mathcal{X} for $i \neq t$ and $1 \leq t \leq n$.

In the following we only need to prove (4) for the case $t = 0$. Set $B = \text{Coker}(C_2 \rightarrow C_1)$. By the induction hypothesis, we get an exact sequence:

$$0 \rightarrow X_n \rightarrow \dots \rightarrow X_3 \rightarrow X_2 \rightarrow C'_1 \rightarrow B \rightarrow 0$$

in \mathcal{A} with C'_1 an object in \mathcal{C} and all X_i objects in \mathcal{X} . Set $K = \text{Coker}(X_3 \rightarrow X_2)$. Then by applying Proposition 5.1(1) to the exact sequence:

$$0 \rightarrow K \rightarrow C'_1 \rightarrow C_0 \rightarrow M \rightarrow 0,$$

we get an exact sequence:

$$0 \rightarrow K \rightarrow X_1 \rightarrow C'_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} with X_1 an object in \mathcal{X} and C'_0 an object in \mathcal{C} . Thus we obtain the desired exact sequence:

$$0 \rightarrow X_n \rightarrow \dots \rightarrow X_2 \rightarrow X_1 \rightarrow C'_0 \rightarrow M \rightarrow 0. \quad \square$$

Let R be a ring. For a module M in $\text{Mod } R$ (resp. $\text{mod } R$), we use $\text{Add}_R M$ (resp. $\text{add}_R M$) to denote the subcategory of $\text{Mod } R$ (resp. $\text{mod } R$) consisting of all modules isomorphic to direct summands of (finite) direct sums of copies of ${}_R M$. Recall that a subcategory \mathcal{D} of $\text{Mod } R$ (resp. $\text{mod } R$) is called *projectively resolving* if \mathcal{D} contains (finitely generated) projective left R -modules, and \mathcal{D} is closed under extensions and kernels of epimorphisms in $\text{Mod } R$ (resp. $\text{mod } R$).

Remark 5.6. Let R and S be rings. Then we have:

(1) Let ${}_R C_S$ be a semidualizing bimodule, that is, the following conditions are satisfied: (c1) ${}_R C$ admits a degreewise finite R -projective resolution, and C_S admits a degreewise finite S^{op} -projective resolution; (c2) both the homothety maps ${}_R R_R \rightarrow \text{Hom}_{S^{\text{op}}}(C, C)$ and ${}_S S_S \rightarrow \text{Hom}_R(C, C)$ are isomorphisms; and (c3) $\text{Ext}_R^i(C, C) = 0 = \text{Ext}_{S^{\text{op}}}^i(C, C)$ for any $i \geq 1$. Then $\mathcal{P}(\text{Mod } R) \cup \text{Add}_R C$ is a generator-cogenerator for the subcategory of $\text{Mod } R$ consisting of G_C -projective modules (see [21] and [17, Corollary 2.10]).

(2) $\mathcal{P}(\text{Mod } R)$ is a projective generator and an injective cogenerator for the subcategory $\mathcal{G}(\text{Mod } R)$ of Gorenstein projective left R -modules. In particular, $\mathcal{G}(\text{Mod } R)$ is projectively resolving by [14, Theorem 2.5].

(3) If R is an Artinian algebra and $T \in \text{mod } R$ is cotilting, then $\text{add}_R(T \oplus R)$ is a generator-cogenerator for ${}^\perp T$ by [2, Theorem 5.4(b)], and clearly ${}^\perp T$ is projectively resolving, where ${}^\perp T = \{M \in \text{mod } R \mid \text{Ext}_R^i(M, T) = 0 \text{ for any } i \geq 1\}$.

(4) Put $\Omega^n(\text{Mod } R) = \{M \in \text{Mod } R \mid \text{there exists an exact sequence } 0 \rightarrow M \rightarrow P_{n-1} \rightarrow P_{n-2} \rightarrow \dots \rightarrow P_0 \text{ in } \text{Mod } R \text{ with all } P_i \text{ projective}\}$ for any $n \geq 1$, and put $\Omega^\infty(\text{Mod } R) = \bigcap_{n \geq 1} \Omega^n(\text{Mod } R)$. Then $\mathcal{P}(\text{Mod } R)$ is a generator-cogenerator for $\Omega^\infty(\text{Mod } R)$. Put $\Omega^n(\text{mod } R) = \Omega^n(\text{Mod } R) \cap \text{mod } R$ for any $n \geq 1$, and put $\Omega^\infty(\text{mod } R) = \bigcap_{n \geq 1} \Omega^n(\text{mod } R)$. Then $\mathcal{P}(\text{mod } R)$ is a generator-cogenerator for $\Omega^\infty(\text{mod } R)$ over a left Noetherian ring R . By [3, Theorem 1.7 and Proposition 2.2], we have that for a left and right Noetherian ring R , if the right flat dimension of the i -th term in a minimal injective coresolution of R_R is at most $i + 1$ for any $i \geq 0$, especially if R is a commutative Gorenstein ring (cf. [5, Fundamental Theorem]), then all $\Omega^n(\text{mod } R)$ and $\Omega^\infty(\text{mod } R)$ are closed under extensions.

(5) A subcategory \mathcal{X} of \mathcal{A} is called *self-orthogonal*, denoted by $\mathcal{X} \perp \mathcal{X}$, if $\text{Ext}_{\mathcal{A}}^i(X, X') = 0$ for any objects X and X' in \mathcal{X} and $i \geq 1$. If $\mathcal{X} \perp \mathcal{X}$, then $\mathcal{G}(\mathcal{X})$ is closed under extensions and \mathcal{X} is a projective generator and an injective cogenerator for $\mathcal{G}(\mathcal{X})$ by [18, Corollaries 4.5 and 4.7]. In the rest of this section, we will focus on the self-orthogonal subcategories of \mathcal{A} .

We write ${}^\perp \mathcal{X}$ (resp. \mathcal{X}^\perp) = $\{M \text{ is an object in } \mathcal{A} \mid \text{Ext}_{\mathcal{A}}^i(M, \mathcal{X}) \text{ (resp. } \text{Ext}_{\mathcal{A}}^i(\mathcal{X}, M)) = 0 \text{ for any } i \geq 1\}$.

Lemma 5.7. Let $\mathcal{X} \perp \mathcal{X}$. Then $\mathcal{G}(\mathcal{X}) = ({}^\perp \mathcal{X} \cap \mathcal{X}^\perp) \cap (\text{res } \widetilde{\mathcal{X}} \cap \text{cores } \widetilde{\mathcal{X}})$.

Proof. It is easy to get the assertion by the definition of $\mathcal{G}(\mathcal{X})$. \square

As an application of Theorem 5.5, we have the following

Theorem 5.8. Let $\mathcal{X} \perp \mathcal{X}$ and M be an object in \mathcal{A} with $\mathcal{G}(\mathcal{X})$ -dim $M < \infty$. Then the following statements are equivalent for any $n \geq 0$.

- (1) $\mathcal{G}(\mathcal{X})$ -dim $M \leq n$.
- (2) There exists an exact sequence:

$$0 \rightarrow C_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} with C_n an object in $\mathcal{G}(\mathcal{X})$ and all X_i objects in \mathcal{X} .

(3) There exists an exact sequence:

$$0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow C_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} with C_0 an object in $\mathcal{G}(\mathcal{X})$ and all X_i objects in \mathcal{X} .

(4) For every non-negative integer t such that $0 \leq t \leq n$, there exists an exact sequence:

$$0 \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} such that X_t is an object in $\mathcal{G}(\mathcal{X})$ and all X_i are objects in \mathcal{X} for $i \neq t$.

(5) $\text{Ext}_{\mathcal{A}}^{n+1}(M, X) = 0$ for any object X in \mathcal{X} and $i \geq 1$.

(6) $\text{Ext}_{\mathcal{A}}^{n+1}(M, Y) = 0$ for any object Y in \mathcal{A} with $\mathcal{X}\text{-dim } Y < \infty$.

(7) $\text{Ext}_{\mathcal{A}}^{n+i}(M, Y) = 0$ for any object Y in \mathcal{A} with $\mathcal{X}\text{-dim } Y < \infty$ and $i \geq 1$.

Proof. Because $\mathcal{X} \perp \mathcal{X}$ by assumption, $\mathcal{G}(\mathcal{X})$ is closed under extensions and \mathcal{X} is a projective generator and an injective cogenerator for $\mathcal{G}(\mathcal{X})$ by [18, Corollaries 4.5 and 4.7]. So by Theorem 5.5, we have (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4).

(7) \Rightarrow (6) and (7) \Rightarrow (5) are trivial.

It is easy to get (1) \Rightarrow (5) \Rightarrow (7) by Lemma 5.7 and the dimension shifting, respectively.

(6) \Rightarrow (1) Let $\mathcal{G}(\mathcal{X})\text{-dim } M = m (< \infty)$. Then by Theorem 5.5, there exists an exact sequence:

$$0 \rightarrow X_m \rightarrow X_{m-1} \rightarrow \cdots \rightarrow X_1 \rightarrow C_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} with all X_i objects in \mathcal{X} and C_0 an object in $\mathcal{G}(\mathcal{X})$. We claim that $m \leq n$. Otherwise, let $m > n$. Note that $\mathcal{X}\text{-dim } \text{Im}(X_{n+1} \rightarrow X_n) \leq m - n - 1 < \infty$. So $\text{Ext}_{\mathcal{A}}^1(\text{Im}(X_n \rightarrow X_{n-1}), \text{Im}(X_{n+1} \rightarrow X_n)) \cong \text{Ext}_{\mathcal{A}}^{n+1}(M, \text{Im}(X_{n+1} \rightarrow X_n)) = 0$ (note: $X_0 = C_0$) by assumption and Lemma 5.7. Hence the exact sequence $0 \rightarrow \text{Im}(X_{n+1} \rightarrow X_n) \rightarrow X_n \rightarrow \text{Im}(X_n \rightarrow X_{n-1}) \rightarrow 0$ splits and $\text{Im}(X_n \rightarrow X_{n-1})$ is isomorphic to a direct summand of X_n . Then by Theorem 4.6(2), $\text{Im}(X_n \rightarrow X_{n-1})$ is an object in $\mathcal{G}(\mathcal{X})$ and $\mathcal{G}(\mathcal{X})\text{-dim } M \leq n$, which is a contradiction. \square

As an immediate consequence of Theorem 5.8, we get the following

Corollary 5.9. Let $\mathcal{X} \perp \mathcal{X}$ and M be an object in \mathcal{A} with $\mathcal{G}(\mathcal{X})\text{-dim } M < \infty$. Then $\mathcal{G}(\mathcal{X})\text{-dim } M = \sup\{n \geq 0 \mid \text{Ext}_{\mathcal{A}}^n(M, X) \neq 0 \text{ for some object } X \text{ in } \mathcal{X}\} = \sup\{n \geq 0 \mid \text{Ext}_{\mathcal{A}}^n(M, Y) \neq 0 \text{ for some object } Y \text{ in } \mathcal{A} \text{ with } \mathcal{X}\text{-dim } Y < \infty\}$.

By Lemma 5.7 and Corollary 5.9, we get the following

Corollary 5.10. Let $\mathcal{X} \perp \mathcal{X}$ and $0 \rightarrow M_3 \rightarrow M_2 \rightarrow M_1 \rightarrow 0$ be an exact sequence in \mathcal{A} with $M_3 \neq 0$ and M_1 an object in $\mathcal{G}(\mathcal{X})$. Then $\mathcal{G}(\mathcal{X})\text{-dim } M_3 = \mathcal{G}(\mathcal{X})\text{-dim } M_2$.

In general, we have $\mathcal{G}(\mathcal{X})\text{-dim } M \leq \mathcal{X}\text{-dim } M$ for any object M in \mathcal{A} . By Corollary 5.9 we get the following

Corollary 5.11. Let $\mathcal{X} \perp \mathcal{X}$ and M be an object in \mathcal{A} with $\mathcal{X}\text{-dim } M < \infty$. Then $\mathcal{G}(\mathcal{X})\text{-dim } M = \mathcal{X}\text{-dim } M$.

Proof. Let $\mathcal{X}\text{-dim } M = n < \infty$. It suffices to prove $\mathcal{G}(\mathcal{X})\text{-dim } M \geq \mathcal{X}\text{-dim } M = n$. It is easy to get that $\text{Ext}_{\mathcal{A}}^n(M, X) \neq 0$ some object X in \mathcal{X} , so $\mathcal{G}(\mathcal{X})\text{-dim } M \geq n$ by Corollary 5.9. \square

By the definition of $\mathcal{G}(\mathcal{X})$, we have that each object in $\mathcal{G}(\mathcal{X})$ can be embedded into an object in \mathcal{X} with the cokernel still in $\mathcal{G}(\mathcal{X})$. The first assertion in the following result generalizes this fact and [7, Lemma 2.17].

Corollary 5.12. *Let $\mathcal{X} \perp \mathcal{X}$ and M be an object in \mathcal{A} and $n \geq 0$, and let $\mathcal{G}(\mathcal{X})\text{-dim } M = n < \infty$. Then we have*

- (1) *There exists an exact sequence $0 \rightarrow M \rightarrow N \rightarrow G \rightarrow 0$ in \mathcal{A} with $\mathcal{X}\text{-dim } N = n$ and G an object in $\mathcal{G}(\mathcal{X})$.*
- (2) *There exists an exact sequence $0 \rightarrow N \rightarrow G \rightarrow M \rightarrow 0$ in \mathcal{A} with $\mathcal{X}\text{-dim } N \leq n - 1$ and G an object in $\mathcal{G}(\mathcal{X})$. This exact sequence is an epic $\mathcal{G}(\mathcal{X})$ -precover of M .*

Proof. Let M be an object in \mathcal{A} with $\mathcal{G}(\mathcal{X})\text{-dim } M = n < \infty$.

(1) We apply Theorem 5.3(1) with $A = 0$ to get an exact sequence $0 \rightarrow M \rightarrow N \rightarrow G \rightarrow 0$ in \mathcal{A} with $\mathcal{X}\text{-dim } N \leq n$ and G an object in $\mathcal{G}(\mathcal{X})$. By Corollary 5.10, we have $\mathcal{G}(\mathcal{X})\text{-dim } N = n$. Then it follows from Corollary 5.11 that $\mathcal{X}\text{-dim } N = n$.

(2) By Theorem 5.8, there exists an exact sequence $0 \rightarrow N \rightarrow G \rightarrow M \rightarrow 0$ in \mathcal{A} with $\mathcal{X}\text{-dim } N \leq n - 1$ and G an object in $\mathcal{G}(\mathcal{X})$. Also by Theorem 5.8, $\text{Ext}_{\mathcal{A}}^1(G', N) = 0$ for any object G' in $\mathcal{G}(\mathcal{X})$. So the above exact sequence is an epic $\mathcal{G}(\mathcal{X})$ -precover of M . \square

Remark 5.13. For an object M in \mathcal{A} , we may define dually the \mathcal{C} -codimension of M , denoted by $\mathcal{C}\text{-codim } M$, as $\inf\{n \geq 0 \mid \text{there exists an exact sequence } 0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \dots \rightarrow C^n \rightarrow 0 \text{ in } \mathcal{A} \text{ with all } C^i \text{ objects in } \mathcal{C}\}$. We set $\mathcal{C}\text{-codim } M$ infinity if no such integer exists. We point out the dual versions on the \mathcal{C} -codimension of all the above results (Theorems 5.5 and 5.8 and Corollaries 5.9–5.12) also hold true by using a completely dual arguments.

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